A FORMULA OF FURSTENBERG-TZKONI TYPE

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ABSTRACT

One establishes a formula for the integral of the $(n+1)$-st powers of the volumes of $(n-1)$-dimensional cross-sections of an ellipsoid that extends a series of formulae given recently by Furstenberg and Tzkoni.

The following formula was found by Furstenberg and Tzkoni [2]:

Let E denote an n-dimensional ellipsoid centered at the origin in R^n and, for $1 \leqq k \leqq n$ let F_k^n denote the Grassman manifold of k-dimensional subspaces of R^n. Put

$$c_{n,k} = \left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^k \Big/ \left\{\frac{k}{2}\Gamma\left(\frac{k}{2}\right)\right\}^n.$$

Then

$$c_{n,k}V_n(E)^k = \int_{F_k^n} V_k(E \cap \zeta)^n dm(\zeta)$$

where $V_n(E)$ denotes the n-dimensional volume of E, $dm(\zeta)$ the normalized rotation invariant measure of F_k^n, and $V_k(E \cap \zeta)$ the k-dimensional volume of the intersection of E and the k-plane ζ through the origin.

The Furstenberg-Tzkoni formula for $k = n-1$ is due to H. Busemann [1].

We denote by $c_n = \pi^{n/2}/(n/2)\Gamma(n/2)$ the volume of the n-dimensional unit ball. The surface area of the elliposid is denoted by $S(E)$. Then

$$\int_{F^n_{n-1}} V_{n-1}(E \cap \zeta)^{n+1}\, dm(\zeta) = \frac{c_{n-1}^{n+1}}{nc_n^n} V_n(E)^{n-1}S(E). \tag{1}$$

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PROOF. We assume that E is the image of the unit sphere under a linear map A of nonvanishing determinant. Let $\sigma(\zeta)$ be the factor by which $(n-1)$-volumes in planes parallel to ζ are multiplied under the action of A. The Gauss curvature K of an hypersurface in R^n is transformed under A into K^A given by ([3], [4] Formula (10))

$$K^A = |\det A|^{n-1}\sigma(\zeta)^{-(n+1)}K,$$

where ζ is the tangent hyperplane to the surface at the point in question. For $(n-1)$-planes in n-space, the invariant measure is the volume element of the projective $(n-1)$-space normalized to a total volume 1; dm is $2/n\, c_n$ times the surface area element of the unit sphere. Hence.

$$\begin{aligned} S(E) &= nc_n \int_{F^n_{n-1}} (K^A)^{-1} dm(\zeta) \\ &= nc_n |\det A|^{-n+1} \int_{F^n_{n-1}} \sigma(\zeta)^{n+1} dm(\zeta) \\ &= \frac{nc_n^n}{c_{n-1}^{n+1} V_n(E)^{n-1}} \int_{F^n_{n-1}} V_{n-1}(E \cap \zeta)^{n+1} dm. \end{aligned}$$

REFERENCES

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